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A $SU(3)$ Wigner function for three-dimensional systems

Alfredo Luis

Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

E-mail: alluis@fis.ucm.es

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Abstract
We derive a Wigner function with $SU(3)$ symmetry for three-dimensional systems exemplified by the polarization of three-dimensional electromagnetic fields.

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1. Introduction

The Wigner-function formulation of quantum mechanics and classical optics is extremely attractive because of the physical insight and the simple formulae that it provides [1–3]. In optics in particular, the Wigner function merges in single-formalism geometrical and wave optics by describing propagation in terms of light rays, including without approximation all second-order coherence phenomena.

In this work we propose a Wigner function for three-dimensional systems (quantum or classical) with $SU(3)$ symmetry. By $SU(3)$ symmetry we mean invariance of basic system properties under unitary 3×3 matrix transformations acting on the three-dimensional vector state. This is not the construction of an analog of $SU(3)$ or of their matrix elements. Instead, this can be regarded as the assembly of nine standard phase-space Wigner functions between pairs of components leading to a single scalar function in an extended space with $SU(3)$ transformation properties.

The main example we have in mind is optical polarization of three-dimensional classical electromagnetic waves [4–6]. (The polarization Wigner function for two-dimensional waves was considered in [3].) In this example, $SU(3)$ transformations correspond to arbitrary energy-conserving linear transformations of the field amplitudes, i.e., the three-dimensional analog of lossless beam splitting. Many other Wigner functions for finite-dimensional systems have been introduced, differing in their symmetries under different transformation groups [7].

2. The $SU(3)$ Wigner function

We focus on three-dimensional systems involving a finite and discrete variable m taking three values, say $m = 1, 2, 3$ without loss of generality. Additionally, some other variables may be necessary to fully describe the system. For definiteness, let us consider unbounded, continuous, Cartesian position variables described by the real vector $\boldsymbol{r}$. Since this is merely an example of additional variables we may consider equally well two- or three-dimensional vectors $\boldsymbol{r}$ without any noticeable change in the following formulae other than the change of $\mathrm{d}^2\boldsymbol{r}$ for two-dimensional $\boldsymbol{r}$ (which may be denoted also as $\boldsymbol{r}_\perp$) by $\mathrm{d}^3\boldsymbol{r}$ for three-dimensional $\boldsymbol{r}$. In classical optics, it is usual to consider field distributions in two-dimensional planes, so that the distribution evolves from the source plane to the observation plane. In any case, the dimension of $\boldsymbol{r}$ is not related to the dimension of the system space. For example, the electric field $\boldsymbol{E}(\boldsymbol{r})$ at each point $\boldsymbol{r}$ is in general a three-dimensional quantity irrespective of the dimension of the distribution of points $\boldsymbol{r}$. Note that we are assuming no transversality condition at work nor any restriction on wave vectors.

The state vector $|\psi\rangle$ for pure, fully coherent states, for example, may be expressed in many forms, such as

$$|\psi\rangle = \begin{pmatrix} \psi_1(\boldsymbol{r}) \\ \psi_2(\boldsymbol{r}) \\ \psi_3(\boldsymbol{r}) \end{pmatrix} \qquad |\psi\rangle = \int \mathrm{d}^2\boldsymbol{r} \sum_{m=1}^{3} \psi_m(\boldsymbol{r})|\boldsymbol{r}\rangle|m\rangle, \tag{2.1}$$

where the vectors $|m\rangle$ represent the m variable and $|\boldsymbol{r}\rangle$ represents position eigenstates with $\langle \boldsymbol{r}'|\boldsymbol{r}\rangle = \delta(\boldsymbol{r}' - \boldsymbol{r})$. In classical optics, $|m\rangle$ are three orthogonal polarization states, and $\psi_m(\boldsymbol{r})$ represent the three electric-field components $E_m(\boldsymbol{r})$ on the polarization basis $|m\rangle$. Since we are assuming no transversality condition nor any restriction on wave vectors, the electric field is a three-dimensional complex vector leading to a continuum of possible polarization states. The electric-field vector can be expressed in different complex three-dimensional bases, where each basis element $|m\rangle$ can always be regarded as representing a definite polarization state. As the simplest example, the three vectors $|m\rangle$ may represent linear vibration (linear polarization) along three mutually orthogonal Cartesian axes, so that any other polarization state is obtained as a complex linear combination of them.

The whole Wigner function including the $\boldsymbol{r}$ and m degrees of freedom can be expressed as

$$W_\rho(\boldsymbol{r}, \boldsymbol{p}, \Omega) = \mathrm{Tr}[\rho \boldsymbol{\Delta}(\boldsymbol{r}, \boldsymbol{p}) \otimes \boldsymbol{\Delta}(\Omega)], \tag{2.2}$$

where $\boldsymbol{p}$ represents linear momentum in quantum mechanics, or the transversal wave vector in classical optics, Ω are the phase-space coordinates required to describe the m variable and we have explicitly used the symbol $\otimes$ for tensor product to emphasize that $\boldsymbol{\Delta}(\boldsymbol{r}, \boldsymbol{p})$ and $\boldsymbol{\Delta}(\Omega)$ act on different spaces. Throughout this paper the upper case trace $\mathrm{Tr}\, A$ means trace on both spaces,

$$\mathrm{Tr}\, A = \sum_{m=1}^{3} \int \mathrm{d}^2\boldsymbol{r} \langle \boldsymbol{r}|\langle m|A|m\rangle|\boldsymbol{r}\rangle \tag{2.3}$$

while the lower case trace $\mathrm{tr}\, B$ will represent trace just on the three-dimensional m-space,

$$\mathrm{tr}\, B = \sum_{m=1}^{3} \langle m|B|m\rangle. \tag{2.4}$$

In quantum mechanics, ρ represents the density matrix while in classical optics it may represent the cross-spectral density matrix $\boldsymbol{\Gamma}$ with the following equivalence:

$$\langle \ell|\langle \boldsymbol{r}_1|\rho|\boldsymbol{r}_2\rangle|m\rangle \propto \Gamma_{\ell,m}(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{2.5}$$

where

$$\Gamma_{\ell,m}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int \mathrm{d}\tau \, \langle E_\ell(\boldsymbol{r}_1, t+\tau) E_m^*(\boldsymbol{r}_2, t)\rangle \exp(\mathrm{i}\omega\tau), \tag{2.6}$$

$\ell, m = 1, 2, 3$, the angle brackets denote ensemble average, and ω is the frequency.

The position-momentum component of the Wigner function $\boldsymbol{\Delta}(\boldsymbol{r}, \boldsymbol{p})$ is well known

$$\boldsymbol{\Delta}(\boldsymbol{r}, \boldsymbol{p}) = \left(\frac{\mu}{2\pi}\right)^2 \int \mathrm{d}^2\boldsymbol{r}' |\boldsymbol{r} + \boldsymbol{r}'/2\rangle\langle \boldsymbol{r} - \boldsymbol{r}'/2| \exp(\mathrm{i}\mu\boldsymbol{p}\cdot\boldsymbol{r}'), \tag{2.7}$$

where $\mu = 1/\hbar$ in quantum mechanics while $\mu = k$ in classical optics, where k is the wavenumber. Note that for classical optics we are considering $\boldsymbol{p}$ as dimensionless. Equivalently, we may say that $\mu\boldsymbol{p}$ is a wave vector $\boldsymbol{k}$ (for three-dimensional $\boldsymbol{r}$ and $\boldsymbol{p}$), or its projection on the plane $\boldsymbol{k}_\perp$ (for two-dimensional $\boldsymbol{r}$ and $\boldsymbol{p}$).

Equivalently,

$$W_\rho(\boldsymbol{r}, \boldsymbol{p}, \Omega) = \mathrm{tr}[\mathbf{H}(\boldsymbol{r}, \boldsymbol{p})\boldsymbol{\Delta}(\Omega)], \tag{2.8}$$

where $\mathbf{H}(\boldsymbol{r}, \boldsymbol{p})$ is the 3×3 Wigner matrix with matrix elements

$$H_{\ell,m}(\boldsymbol{r}, \boldsymbol{p}) = \left(\frac{\mu}{2\pi}\right)^2 \int \mathrm{d}^2\boldsymbol{r}' \langle \ell|\langle \boldsymbol{r} - \boldsymbol{r}'/2|\rho|\boldsymbol{r} + \boldsymbol{r}'/2\rangle|m\rangle \exp(\mathrm{i}\mu\boldsymbol{p}\cdot\boldsymbol{r}'). \tag{2.9}$$

Equation (2.7) is the standard Wigner function for spinless particles in quantum mechanics or for classical light waves. This should be clearly distinguished from another relevant example of Wigner function arising in quantum optics, where the canonical variables $(\boldsymbol{r}, \boldsymbol{p})$ represent field quadratures, (i.e., field components) which have nothing to do with the position momentum or position-wave vector used in (2.7) [1].

Concerning the $\boldsymbol{\Delta}(\Omega)$ component we consider the following family of 3×3 Hermitian matrices:

$$\boldsymbol{\Delta}(\Omega) = \frac{1}{3}\boldsymbol{\Lambda}_0 + \sum_{j=1}^{8} \lambda_j(\Omega)\boldsymbol{\Lambda}_j, \tag{2.10}$$

where $\boldsymbol{\Lambda}_j$ are the nine Gell–Mann matrices (the generators of the $SU(3)$ group) [4]

$$\begin{aligned}
&\boldsymbol{\Lambda}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\boldsymbol{\Lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\Lambda}_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\boldsymbol{\Lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\Lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&\boldsymbol{\Lambda}_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\Lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&\boldsymbol{\Lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \qquad \boldsymbol{\Lambda}_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{2.11}$$

the functions $\lambda_j(\Omega)$ are

$$
\begin{aligned}
\lambda_0 &= 1 \\
\lambda_1 &= 2\sin^2\frac{\theta'}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\phi \\
\lambda_2 &= 2\sin^2\frac{\theta'}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\phi \\
\lambda_3 &= \sin^2\frac{\theta'}{2}\left(\sin^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}\right) \\
\lambda_4 &= 2\cos\frac{\theta'}{2}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\cos(\phi+\phi') \\
\lambda_5 &= 2\cos\frac{\theta'}{2}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\sin(\phi+\phi') \\
\lambda_6 &= 2\cos\frac{\theta'}{2}\sin\frac{\theta'}{2}\cos\frac{\theta}{2}\cos\phi' \\
\lambda_7 &= 2\cos\frac{\theta'}{2}\sin\frac{\theta'}{2}\cos\frac{\theta}{2}\sin\phi' \\
\lambda_8 &= \frac{1}{\sqrt{3}}\left(\sin^2\frac{\theta'}{2}-2\cos^2\frac{\theta'}{2}\right)
\end{aligned}
\tag{2.12}
$$

with $\pi \geqslant \theta, \theta' \geqslant 0, 2\pi \geqslant \phi, \phi' \geqslant 0$. The number of coordinates agrees with the number of parameters required to specify a polarization ellipse in a three-dimensional space: two angles are necessary to specify the plane containing the ellipse, and two more angles determine the eccentricity and orientation of the ellipse.

The matrices $\mathbf{\Lambda}_j$ and the functions $\lambda_j(\Omega)$ satisfy similar orthogonality relations

$$
\begin{aligned}
&\operatorname{tr}(\mathbf{\Lambda}_j\mathbf{\Lambda}_\ell) = 2\delta_{j,\ell} + \delta_{j,0}\delta_{\ell,0} \\
&\int \mathrm{d}\Omega\,\lambda_j(\Omega)\lambda_\ell(\Omega) = \frac{(4\pi)^2}{6}(\delta_{j,\ell} + 5\delta_{j,0}\delta_{\ell,0})
\end{aligned}
\tag{2.13}
$$

with

$$
\mathrm{d}\Omega = 4\sin\theta\cos\frac{\theta'}{2}\sin^3\frac{\theta'}{2}\,\mathrm{d}\theta\,\mathrm{d}\theta'\,\mathrm{d}\phi\,\mathrm{d}\phi' \tag{2.14}
$$

and

$$
\int \mathrm{d}\Omega = (4\pi)^2. \tag{2.15}
$$

Alternatively, the family $\mathbf{\Delta}(\Omega)$ can be expressed as

$$
\mathbf{\Delta}(\Omega) = 2\boldsymbol{\mathcal{E}}(\Omega)\boldsymbol{\mathcal{E}}^\dagger(\Omega) - \tfrac{1}{3}\mathbf{\Lambda}_0, \tag{2.16}
$$

where $\boldsymbol{\mathcal{E}}(\Omega)$ are the states obtained by the action of an $SU(3)$ transformation $\mathbf{V}(\Omega)$ on the vector (0, 0, 1) [5, 8],

$$
\boldsymbol{\mathcal{E}}(\Omega) = \begin{pmatrix} \sin\frac{\theta}{2}\sin\frac{\theta'}{2}\exp[-\mathrm{i}(\phi+\phi')] \\ \cos\frac{\theta}{2}\sin\frac{\theta'}{2}\exp(-\mathrm{i}\phi') \\ \cos\frac{\theta'}{2} \end{pmatrix} = \mathbf{V}(\Omega)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.17}
$$

with $\boldsymbol{\mathcal{E}}^\dagger(\Omega)\boldsymbol{\mathcal{E}}(\Omega) = 1$, and $\boldsymbol{\mathcal{E}}^\dagger(\Omega)\mathbf{\Lambda}_j\boldsymbol{\mathcal{E}}(\Omega) = \lambda_j(\Omega)$.

3. Main properties

The Wigner function proposed above has the following properties:

(i) The family of operators $\mathbf{\Delta}(\Omega)$ provides an invertible correspondence $\mathbf{A} \leftrightarrow W_{\mathbf{A}}(\Omega)$ between arbitrary 3×3 matrices $\mathbf{A}$ and functions $W_{\mathbf{A}}(\Omega)$:

$$W_{\mathbf{A}}(\Omega) = \mathrm{tr}[\mathbf{A}\mathbf{\Delta}(\Omega)] \qquad \mathbf{A} = \frac{3}{(4\pi)^2}\int \mathrm{d}\Omega\, W_{\mathbf{A}}(\Omega)\mathbf{\Delta}(\Omega). \tag{3.1}$$

(ii) $W_\rho(\boldsymbol{r}, \boldsymbol{p}, \Omega)$ provides complete information about second-order properties of the system (i.e., proportional to $\langle \psi_\ell \psi_m^* \rangle$ or $\langle E_\ell E_m^* \rangle$) since we can express the whole density matrix (or cross-spectral density tensor) in terms of $W_\rho(\boldsymbol{r}, \boldsymbol{p}, \Omega)$:

$$\langle \ell | \langle \boldsymbol{r}_1 | \rho | \boldsymbol{r}_2 \rangle | m \rangle = \frac{3}{(4\pi)^2}\int \mathrm{d}^2\boldsymbol{p}\, \mathrm{d}\Omega\, W_\rho[(\boldsymbol{r}_1 + \boldsymbol{r}_1)/2, \boldsymbol{p}, \Omega] \exp[\mathrm{i}\mu \boldsymbol{p}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)] \times \langle \ell | \mathbf{\Delta}(\Omega) | m \rangle. \tag{3.2}$$

(iii) By construction $\mathbf{\Delta}^\dagger(\Omega) = \mathbf{\Delta}(\Omega)$ so that matrix Hermitian conjugation is equivalent to complex conjugation of the Wigner function $W_{\mathbf{A}^\dagger}(\Omega) = W^*_{\mathbf{A}}(\Omega)$ and vice versa. In particular, Hermitian matrices $\mathbf{A} = \mathbf{A}^\dagger$ (such as density matrices and cross-spectral density tensors) are associated with real functions $W_{\mathbf{A}}(\Omega) = W^*_{\mathbf{A}}(\Omega)$.

(iv) Matrix trace corresponds to Ω integration:

$$\begin{aligned} &\frac{3}{(4\pi)^2}\int \mathrm{d}\Omega\, W_{\mathbf{A}}(\Omega) = \mathrm{tr}\,\mathbf{A} \\ &\frac{3}{(4\pi)^2}\int \mathrm{d}\Omega\, W_{\mathbf{A}}(\Omega) W_{\mathbf{B}}(\Omega) = \mathrm{tr}(\mathbf{AB}). \end{aligned} \tag{3.3}$$

(v) The correspondence $\mathbf{A} \leftrightarrow W_{\mathbf{A}}(\Omega)$ is endowed with a suitable transformation law under $SU(3)$ transformations $\mathbf{U}$:

$$W_{\mathbf{U}^\dagger\mathbf{A}\mathbf{U}}(\Omega) = W_{\mathbf{A}}(\Omega'), \tag{3.4}$$

where $\Omega' = \Omega'(\Omega, \mathbf{U})$ is a coordinate transformation with $\mathrm{d}\Omega' = \mathrm{d}\Omega$ [5]. This is because

$$W_{\mathbf{U}^\dagger\mathbf{A}\mathbf{U}}(\Omega) = \mathrm{tr}[\mathbf{A}\mathbf{U}\mathbf{\Delta}(\Omega)\mathbf{U}^\dagger] \tag{3.5}$$

and

$$\mathbf{U}\mathbf{\Delta}(\Omega)\mathbf{U}^\dagger = 2\mathbf{U}\boldsymbol{\mathcal{E}}(\Omega)\boldsymbol{\mathcal{E}}^\dagger(\Omega)\mathbf{U}^\dagger - \tfrac{1}{3}\mathbf{\Lambda}_0 = 2\boldsymbol{\mathcal{E}}(\Omega')\boldsymbol{\mathcal{E}}^\dagger(\Omega') - \tfrac{1}{3}\mathbf{\Lambda}_0, \tag{3.6}$$

where

$$\boldsymbol{\mathcal{E}}(\Omega') = \mathbf{U}\boldsymbol{\mathcal{E}}(\Omega) = \mathbf{U}\mathbf{V}(\Omega)\begin{pmatrix}0\\0\\1\end{pmatrix} = \mathbf{V}(\Omega')\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{3.7}$$

Therefore $\mathbf{U}\mathbf{\Delta}(\Omega)\mathbf{U}^\dagger = \mathbf{\Delta}(\Omega')$ and the $SU(3)$ transformation $\mathbf{U}^\dagger\mathbf{A}\mathbf{U}$ is equivalent to the transformation of the arguments of $W_{\mathbf{A}}(\Omega)$.

The practical implementation of these transformations is not as simple as in the $SU(2)$ case. Nevertheless, we can take advantage of the fact that $SU(3)$ transformations can be expressed as the product of three consecutive $SU(2)$ transformations between pairs of components [8], so they might be implemented by the combination of beam splitters and phase shifters. This has already been taken into account to construct arbitrary $SU(n)$ transformations in a slightly different context in [9].

(vi) The $SU(3)$ Wigner function can be related to the normalized $SU(3)$ Q function defined by projection on the states $\mathcal{E}(\Omega)$ as

$$Q_{\mathbf{A}}(\Omega) = \frac{3}{(4\pi)^2}\mathcal{E}^{\dagger}(\Omega)\mathbf{A}\mathcal{E}(\Omega) \tag{3.8}$$

so that

$$W_{\mathbf{A}}(\Omega) = \tfrac{2}{3}(4\pi)^2 Q_{\mathbf{A}}(\Omega) - \tfrac{1}{3}\operatorname{tr}\mathbf{A}. \tag{3.9}$$

(vii) An alternative expression for $W(\boldsymbol{r}, \boldsymbol{p}, \Omega)$ is

$$W(\boldsymbol{r}, \boldsymbol{p}, \Omega) = \frac{1}{3}S_0(\boldsymbol{r}, \boldsymbol{p}) + \sum_{j=1}^{8}\lambda_j(\Omega)S_j(\boldsymbol{r}, \boldsymbol{p}), \tag{3.10}$$

where $S_j(\boldsymbol{r}, \boldsymbol{p})$, for $j = 0, 1, \ldots, 8$, are

$$S_j(\boldsymbol{r}, \boldsymbol{p}) = \operatorname{Tr}[\rho\boldsymbol{\Delta}(\boldsymbol{r}, \boldsymbol{p}) \otimes \boldsymbol{\Lambda}_j] = \operatorname{tr}[\mathbf{H}(\boldsymbol{r}, \boldsymbol{p})\boldsymbol{\Lambda}_j]. \tag{3.11}$$

In classical optics the functions $S_j(\boldsymbol{r}, \boldsymbol{p})$ can be regarded as generalized Stokes parameters of the light ray at point $\boldsymbol{r}$ propagating along the direction specified by $\boldsymbol{p}$ [3].

(viii) Focusing on classical optics we can construct a reduced space-polarization Wigner function $W(\boldsymbol{r}, \Omega)$ by removing the dependence on $\boldsymbol{p}$,

$$W(\boldsymbol{r}, \Omega) = \int \mathrm{d}^2\boldsymbol{p}\, W(\boldsymbol{r}, \boldsymbol{p}, \Omega) = \operatorname{tr}[\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r})\boldsymbol{\Delta}(\Omega)] \tag{3.12}$$

which describes the polarization properties at point $\boldsymbol{r}$. This is equivalent to

$$W(\boldsymbol{r}, \Omega) = \frac{1}{3}s_0(\boldsymbol{r}) + \sum_{j=1}^{8}\lambda_j(\Omega)s_j(\boldsymbol{r}), \tag{3.13}$$

where

$$s_j(\boldsymbol{r}) = \int \mathrm{d}^2\boldsymbol{p}\, S_j(\boldsymbol{r}, \boldsymbol{p}) = \operatorname{tr}[\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r})\boldsymbol{\Lambda}_j] \tag{3.14}$$

are the generalization to three-dimensional electromagnetic fields of the standard Stokes parameters [4, 5].

(ix) The reduced space-polarization Wigner function allows us to express the degree of polarization $P(\boldsymbol{r})$ for the three-dimensional light at point $\boldsymbol{r}$ as

$$P^2(\boldsymbol{r}) = 8\pi^2 \int \mathrm{d}\Omega \left[W_N(\boldsymbol{r}, \Omega) - \frac{1}{(4\pi)^2}\right]^2. \tag{3.15}$$

This is the distance between the normalized Wigner function associated with $\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r})$:

$$W_N(\boldsymbol{r}, \Omega) = \frac{W(\boldsymbol{r}, \Omega)}{\int \mathrm{d}\Omega'\, W(\boldsymbol{r}, \Omega')} \tag{3.16}$$

and the uniform Wigner function $W_N(\boldsymbol{r}, \Omega) = 1/(4\pi)^2$ associated with second-order fully unpolarized light with $\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r}) \propto \boldsymbol{\Lambda}_0$. By property (v) above the degree of polarization is invariant under $SU(3)$ transformations.

The definition (3.15) coincides with previous approaches to the degree of polarization $P(\boldsymbol{r})$ for three-dimensional light, which are derived from the Hilbert–Schmidt matrix distance to unpolarized light, or as the length of the Stokes vector [4–6],

$$P^2(\boldsymbol{r}) = \frac{3}{2}\operatorname{tr}\left[\left(\frac{1}{3}\boldsymbol{\Lambda}_0 - \frac{\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r})}{\operatorname{tr}\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r})}\right)^2\right] = \frac{3}{4}\frac{\sum_{j=1}^{8} s_j^2(\boldsymbol{r})}{s_0^2(\boldsymbol{r})}. \tag{3.17}$$

Finally, $P(\boldsymbol{r})$ can also be expressed as

$$P^2(\boldsymbol{r}) = 32\pi^2 \int \mathrm{d}\Omega \left[Q_\gamma(\boldsymbol{r}, \Omega) - \frac{1}{(4\pi)^2} \right]^2. \tag{3.18}$$

This is the distance between the Q function Q_γ associated with the normalized cross-spectral density tensor $\gamma(\boldsymbol{r}) = \boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r}) / \operatorname{tr}[\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r})]$ and the uniform Q function $Q_{\Lambda_0}(\boldsymbol{r}, \Omega) = 1/(4\pi)^2$ associated with second-order fully unpolarized light with $\boldsymbol{\Gamma}(\boldsymbol{r}, \boldsymbol{r}) \propto \boldsymbol{\Lambda}_0$. This idea of degree of polarization as a distance between Q functions agrees with the analyses in [5, 10].

4. Conclusions

We have derived a scalar, real and complete Wigner function for three-dimensional systems with $SU(3)$ symmetry, classical and quantum. The action of unitary 3×3 matrices is simply described by a change of variables. The $SU(3)$ Wigner function can be alternatively expressed in terms of generalized Stokes parameters. In particular, we have shown that this provides a simple picture of the degree of polarization of three-dimensional electromagnetic fields.

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